Biospectra : Vol. 8(2), September, 2013, Spl. issue. pp 81-84
An International Biannual Refereed Journal of Life Sciences

Interdisciplinary Sciences
Abst. No. - MBC 03

ISSN : 0973-7057

A mathematical model of a lentic water body, Kota, Rajasthan

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Received 25th June, 2013; Revised 30th July, 2013

Abstract : Mathematical and statistical models can serve as tools for understanding the epidemiology of human immunodeficiency virus and acquired immunodeficiency syndrome if they are constructed carefully. A mathematical model has been developed for the formulation and simulation of real phenomenon by which predictions and forecasts can be made. These models developed using differential equations. The full dynamics of the model and the effects of varying the functional forms are not completely understood. Moreover, the effects of small-scale physical influences are only recently becoming apparent. We investigated the use of a simple plankton population model based on observation from a pond.

. **Key words: : Pond, nutrients, phyto plankton, mathematical model, population size, quantity of resources, Rajasthan.**

INTRODUCTION

The prospect of climate change has stimulated research into several biological processes that might affect climate. One such process that has attracted is change in temperature of water. In natural water bodies, interactions amongst various components continued endlessly. Most common functional interaction of such water bodies are food interactions. Phytoplanktons depend on nutrients present in water. Simple models of plankton-nutrient populations often consist of ordinary differential equations, describing the time dependence of nutrients and Phytoplanktons in aquatic ponds.

BACKGROUND

Deterministic mathematical models of nutrient-plankton interaction with different complexity have been constructed and analyzed by Riley *et al.* (1949). The majority of these models that appeared later were formulated in terms of differential equations (Steele and Henderson 1981, 1992; Evans and Parslow, 1985; Busenberg *et al.* 1990; Edwards , 2001; Jang and Baglama, 2000, 2003). Ruan (2001) analyzed the Oscillations in various plankton models along with nutrient recycling. Edwards (2001) discussed a dynamic approach while adding detritus to classical NPZ model. In the following account a simple model is derived on change of rate of Nutrients and amount of nutrients. The conditions of stability or non-stability are also discussed.

Section 1

The model:

The present model is based on the assumption that the nutrients and planktons are in an equilibrium state. These are well distributed and also there is no drift in their concentration and numbers respectively. While constructing this model ordinary differential equations were used.

$$dN/dt = A - \alpha_o N$$

Here, A is a constant, N represents concentration of nutrient and t is time dN/dt is change in concentration of nutrients over time.

In the next part of paper, a simple model for

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In the next part of paper, a simple model for interaction of Nutrients and Plankton population is framed and stability analysis and hopf-bifurcation is described.

Section 2

Consider the system when only nutrients are present, hence the equation will be:

$dN / dt = A - \alpha_o N$ (1.1)

N represents concentration of nutrient and t is time.

dN/dt is change in concentration of nutrients over time.

$N = (n_1, n_2, n_3, \ldots\ldots n_n)$

Definition

Let the $N = \varphi(t)$ is solution of (1.1), defined for all $t \geq 0$. The solution φ is said to be stable if for every € □ 0 there is a δ □ 0 such that if □ is any solution of (1.1) satisfying | □ (0) - φ (0) |□ δ then □ is defined for all t □0 and | □ (t) - φ (t) | □ € for all t □ 0. If φ is not stable it is said to be unstable.

For non linear equations it is not always possible to answer the equation for stability or instability for all solutions of equation (1.1). For linear equations with constant coefficients the question can always be solved. For linear system we have following theorem:

Theorem 1.1

Every solution of $dN/dt = A - \alpha_o N$ is stable if and only if the identically zero solution is stable.

The Proof:

Let φ (t) be the fundamental matrix associated with (1.1), that is $\varphi'(t) = A^*\varphi(t)$ and $\varphi(0) = I$, the identity matrix. Any solution □of (1.1) may be written in the form □(t) = c φ(t) for some constant vector c, where c = □(0). So, first we assume that the identically zero solution is stable we get, that for every € □ 0 there is a δ□ 0 such that if | □ (0) - 0 | □ δ, then | □ (t) - 0 | □ €. i.e. if | c |□ δ then | c φ(t) | □ €. (1.2).

Now, let φ be an arbitrary solution of (1.1) and let $\varphi_1(0)$ = □.

If | □ (0) - φ_1 (0) |□ δ then | c - □ | □ δ . So, replace c in (1.2) by c – □.

we conclude then that |(c – □) φ (t) | □ €. Whenever | c - □ | □ δ.

But (c–□) φ (t) = c φ (t) - □ φ (t) = □ (t) – φ_1(t) (1.3)

So we have | □ (t) - φ_1 (t) |□ € whenever | □ (0) - φ_1 (0) | □ δ.

Thus, φ_1 is stable.

Conversely if every solution is stable, then certainly zero solution is stable. So, the theorem is proved.

Again N = φ (t) is said to be asymptotically stable if for every € □ 0 there is a δ □ 0 such that if □ is any solution of (1.1) satisfying

| □(0) - φ_1 (0) | □ δ then □ is defined for all t□0;

| □ (t) - φ_1 (t) | □ € .

$\lim_{t \to \infty}$ | □ (t) - φ_1 (t) | = 0 (1.4)

The zero solution of (1.1) is unstable if the real part of atleast of the eigen values of A^* is positive and it is asymptotically stable if the real parts of eigen values of A^* are negative.

$dN/dt = d\xi/dt = f(N) = f(N_0 + \xi) = g(\xi)$

$d\xi/dt = A^* \xi$ (1.5)

We shall require A* such that

$|g(\xi) - A^* \xi|$ is very small.

$\lim_{|\xi| \to 0} |g(\xi) - A^* \xi| / |\xi| = 0$ (1.6)

The components of vector g (ξ) are differentiable.

Section 3

Consider the system for interaction between nutrients and Phytoplanktons:-

$dN / dt = A - \alpha_1 PN +$ □ $P - \alpha_o N$ (1a)

$dP/dt = \beta_0 \alpha_1 PN - \beta_1 P - \beta_2 P^2$ (1b)

Where dN/dt represents rate of change of concentrations of nutrients,

dP/dt represents rate of change concentrations of Phytoplanktons

α_1, □ , α_0, β_0, β_1, β_2 are positive parameters.

0□β_0□ 1; β_1 □ □

α_0 = rate of nutrients flushed out.

α_1 = rate of up taking nutrients by Phytoplanktons.

□ = rate of conversion of Phytoplanktons into nutrients.

A = constant input through human action.

β_0 = rate of consumption of nutrients by Phytoplanktons

β_1 = phytoplankton washout rate.

β_2 = death rate of Phytoplanktons.

<u>**Case I**</u>

Trivial equilibrium point E0 (0, 0) always exist.

<u>**Case II**</u>

The equilibrium point E_1 (A/α_0, 0) exist on boundary.

<u>**Case III**</u>

Non-trivial equilibrium E_2 (N*, P*) exist if there is a positive solution exists to the following set of equations:

f_1 (N, P) a" $A - \alpha_1 PN +$ □ $P - \alpha_o N = 0$ (2a)

f_2 (N, P) a" $\beta 0\, \alpha_1 PN - \beta_1 P - \beta_{2\,P2} = 0$ (2b)

$$N^* = \frac{\beta 1 + \beta 2}{\beta 0 \alpha 1}$$

P* is given by the quadratic equation

$\alpha_1 \beta_2 P^{*2} + (\alpha_1 \beta_1 + \alpha_0 \beta_2 - \alpha_1 u \beta_0) P^* + (\alpha_0 \beta_1 - A \alpha_1 \beta_0) = 0$

Variational matrix is given by:

$$V = \begin{bmatrix} \alpha 1 P - \alpha 0 & \cdots & -\alpha 1 N \\ \vdots & \ddots & \vdots \\ \beta 0 \alpha 1 P & \cdots & \beta 0 \alpha 1 N - \beta 1 - 2\beta 2 P \end{bmatrix} \quad (3)$$

At point E_0

$$V_0 = \begin{bmatrix} -\alpha 0 & \cdots & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\beta 1 \end{bmatrix} \quad (4)$$

Eigen values of which are $-\alpha_o$ and $-\beta_1$; both are negative. Hence (0,0) is stable node.

At point E_1 :

$$V_1 = \begin{bmatrix} -\alpha 0 & \cdots & -\alpha 1 \frac{A}{\alpha 0} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta 0 \alpha 1 \frac{A}{\alpha 0} - \beta 1 \end{bmatrix} \quad (5)$$

Eigen values of which are $-\alpha_o$ and $\beta_0 \alpha_1 \frac{A}{\alpha c} - \beta_1$. If $\beta_1 < \beta_0 \alpha_1$; we have a saddle point . Hence the system is unstable.

At point E2:

$$V_2 = \begin{bmatrix} \alpha 1 P * - \alpha 0 & \cdots & -\alpha 1 N * \\ \vdots & \ddots & \vdots \\ \beta 0 \alpha 1 P * & \cdots & \beta 0 \alpha 1 N * - \beta 1 - 2\beta 2 P * \end{bmatrix} \quad (6)$$

Eigen values are given by: $\lambda i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Where:

$b = \alpha_1 P^* - \alpha_0 - \beta_2 P^*$

$a = 1$

$c = \alpha_{12} P^* (\beta_1 + \beta_2 P^*) - \alpha_1 \beta_2 P^{*2} - \alpha_1 \square \beta_0 P^* - \alpha_0 \beta_2 P^*$

If the sign of the Eigen values above will differ, then fixed point is the saddle point. The system, hence, is unstable. The stability of this fixed point is of importence. If it were stable non-zero populations might be attracted towards it. The dynamics of the system might lead towards the extinction of both nutrients and Phytoplanktons for many cases of initial population level. If the sign of Eigen values above remains same the system is stable.

DISCUSSION

The model described and discussed above is a simplified model. In this model simplified version of the system has been depicted. In the present paper the analytical method disclosed that even if the system is in a steady state and if the fluctuations of the medium occur, one can predict about the system. We also found that the plankton dominance change according to the time. Thus, we found that as long as assumptions (well incorporated planktons and absence of intra-trophism) are applicable, the model serves the needs of the biological predictions.

REFERENCES

1. **Busenberg, S., Kumar, S.K., Austin, P. and Wake, G. 1990**. The dynamics of a model of a plankton-nutrient interaction. *Bull. Mat. Bil.* **52**, 677-696.
2. **Edwards, A.M.2001.** Adding detritus to a Nutrient-phytoplankton-zooplankton model; a dynamical-system approach. *J. Plankton Res.,* **23,** 389-413.
3. **Evans, G.T. and Parslow, J.S. 1985.** A model of annual plankton cycles. B*ull. Oceanogr.*, **3**, 327-347, 1985.
4. **Jang, S.R.J., and Baglama, J. 2000.** Qualitative behavior of a variable-yield simple food chain with an inhibiting nutrient. *Math. Biosci.* **164,** 65-80, 2000.
5. **Jang, S.R.J., and Baglama, J. 2003.** Persistence in variable-yield nutrient plankton models. *Math. Comput. Modelling,* ***38***, 281-298, 2003.
6. **Riley, G.A. , Stommel, H., Burrpus, D.P. 1949.** Qualitative ecology of the plankton of the Western North Atlantic. *Bull. Bingham Oceanogr. Collect*. **12**, 1-169.
7. **Ruan, S. 2001.** Oscillations in plankton models with nutrient cycling. *J. Theor. Biol*. **208**, 15-26, 2001.
8. **Steele, J.H., and Henderson E.W. 1981**. A simple plankton model. *Am Nat.* **117**, 676-691.
9. **Steele, J.H., and Henderson E.W. 1992.** The role of predation in plankton models. *J. Plankton Res.* **14**, 157-72.
10. **Theory Of Ordinary Differential Equations by Codding and Levinson**.
11. **Discrete and Continuous Methods in Applied Mathematics by Jercold C. Methews and Carl E. Langenhog.**

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